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FAST TRACK COMMUNICATION

Transmission–reflection problem with a potential of the form of the derivative of the delta function

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Abstract
Regarding the quantum mechanical transmission–reflection problem in one dimension with a potential of the form of the derivative of the Dirac delta function $\delta'(x) = \mathrm{d}\delta(x)/\mathrm{d}x$, Christiansen *et al* recently found that, depending on how $\delta'(x)$ is interpreted, there can be a resonance which leads to partial transmission. This is in contrast to the earlier consensus that such a potential allows no transmission. The $\delta'(x)$ can be regarded as the narrow-width limit of a certain function $\Delta'(x)$ of a finite range. Christiansen *et al* assumed a rectangular function for $\Delta'(x)$. We examine various other forms and how the resonance depends on the shape of $\Delta'(x)$. We also present some general observations related to the 'threshold anomaly'.

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The transmission–reflection problem in one-dimensional quantum mechanics with a potential of the form of

$$V(x) = \frac{(\hbar\sigma)^2}{2m}\delta'(x) \tag{1}$$

is a subject of considerable interest. Here m is the mass of the particle under consideration, σ is a dimensionless constant, $\delta(x)$ is the Dirac delta function and $\delta'(x) = \mathrm{d}\delta(x)/\mathrm{d}x$. In the following we use units such that $\hbar^2/(2m) = 1$. Until recently there was a consensus that potential (1) allows no transmission, i.e., the wave incident on the potential site is totally reflected at all energies [1–3]. Christiansen *et al*, however, recently found that, depending on how the $\delta'(x)$ is interpreted, there can be a resonance or a bound state at threshold and it leads to partial transmission [4]. The $\delta'(x)$ can be regarded as the narrow-width limit of a $\Delta'(x)$ that has a finite width. It is understood that

$$\Delta(x) = \int_{-\infty}^{x} \Delta'(y)\,\mathrm{d}y, \qquad \int_{-\infty}^{\infty} \Delta(x)\,\mathrm{d}x = 1. \tag{2}$$

The $\Delta(x)$ is a regularized delta function of a finite range. In [4] a specific rectangular function was assumed for $\Delta'(x)$. The purpose of this communication is to examine various other forms of $\Delta'(x)$ and show how the resonance depends on the shape of $\Delta'(x)$.

Let us summarize the background of this problem. Consider the situation in which a particle of energy E is incident from the left (the negative x side). The associated incident wave is e^{ikx}, where $k > 0$ is related to the energy by $E = k^2$. The wavefunction can be written as

$$\psi(x) = \begin{cases} e^{ikx} + R(k)\, e^{-ikx} & \text{for} \quad x < 0 \\ T(k)\, e^{ikx} & \text{for} \quad x > 0, \end{cases} \tag{3}$$

where $T(k)$ and $R(k)$ are the transmission and reflection coefficients, respectively. The probabilities of transmission and reflection are respectively given by $|T(k)|^2$ and $|R(k)|^2$. The wavefunction is determined by the Schrödinger equation

$$-\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x). \tag{4}$$

We solve it with a potential

$$V(x) = \sigma^2 \Delta'(x). \tag{5}$$

Then we take the narrow width limit of $\Delta'(x)$.

We consider various versions of $\Delta'(x)$ which we denote with $\Delta'_i(x)$, $i = 1, 2, \ldots$, starting with the dipole form that was assumed in [1–3],

$$\Delta'_1(x) = \frac{1}{\epsilon}\left[\delta\left(x + \frac{\epsilon}{2}\right) - \delta\left(x - \frac{\epsilon}{2}\right)\right], \tag{6}$$

where $\epsilon > 0$. This is the derivative of the rectangular function

$$\Delta_1(x) = \begin{cases} 1/\epsilon & \text{if} \quad |x| < \epsilon/2 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Potential (5) with $\Delta'_1(x)$ leads to, in the limit of $\epsilon \to 0$,

$$\psi(0) = T(k) = 1 + R(k) = 0. \tag{8}$$

There is no transmission at any energies and the incident wave is totally reflected at $x = 0$. This means that the two half spaces of $x > 0$ and $x < 0$ become effectively disjoint [1–3].

Christiansen *et al* [4] considered the following version of $\Delta'(x)$:

$$\Delta'_{\epsilon,l}(x) = \begin{cases} \pm 1/(\epsilon l) & \text{if} \quad -(\epsilon \pm l)/2 < x < (\epsilon \mp l)/2 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

They examined the following two different limiting processes for $\Delta'_{\epsilon,l}(x)$:

$$\lim_{\epsilon\to 0}\lim_{l\to\epsilon} \Delta'_{\epsilon,l}(x), \tag{10}$$

and

$$\lim_{l\to 0}\lim_{\epsilon\to 0} \Delta'_{\epsilon,l}(x). \tag{11}$$

It is understood that such limits are always taken after the Schrödinger equation has been solved. The result of process (11) is the same as that of (6). So we do not discuss it any more. With the more interesting (10) we have

$$\Delta'_2(x) = \begin{cases} 0 & \text{if} \quad |x| > \epsilon \\ -(x/|x|)(\epsilon)^{-2} & \text{if} \quad |x| < \epsilon, \end{cases} \tag{12}$$

which is the derivative of a triangular function of a unit area. For this $\Delta'_2(x)$ Christiansen *et al* found that, if the strength parameter σ satisfies

$$\tan\sigma = \tanh\sigma, \tag{13}$$

the potential becomes partially transparent. Equation (13) admits, in addition to the trivial solution $\sigma_0 = 0$, a series of solutions σ_n with $n = 1, 2, \ldots$. For these solutions the transmission probability is given by

$$|T(k)|^2 = (\sec\sigma\,\mathrm{sech}\,\sigma)^2 = 1 - \tanh^4\sigma. \tag{14}$$

The transmission probability obtained in this way is energy independent. This ends our summary of the background of the problem.

For $\Delta'_{\epsilon,l}(x)$ of (9) let us assume that l is related to ϵ by $l = \lambda\epsilon$, where $\lambda \geqslant 1$ is a dimensionless factor that is kept fixed in the limiting process of $\epsilon \to 0$. We indicate the $\Delta'(x)$ defined in this way with index $i = 3$, i.e.,

$$\Delta'_3(x) = \begin{cases} \pm 1/(\lambda\epsilon^2) & \text{if} \quad -(1\pm\lambda)\epsilon/2 < x < (1\mp\lambda)\epsilon/2 \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

The $\Delta'_2(x)$ is a special case of $\Delta'_3(x)$ with $\lambda = 1$. The height and depth of the potential are proportional to $1/\epsilon^2$. We solve the transmission–reflection problem with potential $V(x) = \sigma^2\Delta'_3(x)$ and obtain

$$T(k) = \frac{2\mathrm{e}^{-2\mathrm{i}\eta}}{D_\lambda}, \qquad \eta = k\epsilon, \tag{16}$$

where

$$D_\lambda = D_{\lambda=1} - \frac{1}{2}[\mathrm{e}^{2\mathrm{i}(\lambda-1)\eta} - 1]\left(\frac{\alpha}{\eta} - \frac{\eta}{\alpha}\right)\left(\frac{\beta}{\eta} + \frac{\eta}{\beta}\right)\sin\alpha\sinh\beta, \tag{17}$$

$$\begin{aligned} D_{\lambda=1} = {} & 2\cos\alpha\cosh\beta - \left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)\sin\alpha\sinh\beta \\ & - \mathrm{i}\left[\left(\frac{\alpha}{\eta} + \frac{\eta}{\alpha}\right)\sin\alpha\cosh\beta - \left(\frac{\beta}{\eta} - \frac{\eta}{\beta}\right)\cos\alpha\sinh\beta\right], \end{aligned} \tag{18}$$

$$\alpha = \sqrt{\tau^2 + \eta^2}, \qquad \beta = \sqrt{\tau^2 - \eta^2}, \qquad \tau = \sigma/\sqrt{\lambda}. \tag{19}$$

The $D_{\lambda=1}$ is the same as the D of [4]. Note that $T(k)$ is expressed in terms of σ, λ and $\eta = k\epsilon$, which are all dimensionless. This is a consequence of the scaling feature of $\Delta'_3(x) \propto 1/\epsilon^2$. We eventually let $\epsilon \to 0$ and $\eta \to 0$. In this limit $T(k)$ becomes a function of σ and λ, which are independent of k.

When $\eta \ll 1$, D_λ can be expanded as

$$D_\lambda = \cos\tau\cosh\tau\left[-\frac{\mathrm{i}\tau}{\eta}f(\tau,\lambda) + 2 + (\lambda-1)^2\tau^2\tan\tau\tanh\tau\right] + \mathrm{O}(\eta), \tag{20}$$

where $\mathrm{O}(\eta)$ is of the order of η and

$$f(\tau,\lambda) = \tan\tau - \tanh\tau + (\lambda-1)\tau\tan\tau\tanh\tau. \tag{21}$$

Unless $f(\tau,\lambda) = 0$, we obtain $D_\lambda \to \infty$ and $T(k) \to 0$ as $\eta \to 0$. That is, no transmission takes place. If $f(\tau,\lambda) = 0$, however, the potential becomes partially transparent. Equation $f(\tau,\lambda) = 0$ admits, in addition to the trivial solution $\tau_0 = \sigma_0 = 0$, a series of solutions $\tau_n = \sigma_n/\sqrt{\lambda}$ with $n = 1, 2, \ldots$. For these solutions the transmission probability is given by

$$|T(k)|^2 = (\sec\tau\,\mathrm{sech}\,\tau)^2\left[1 + \tfrac{1}{2}(\lambda-1)^2\tau^2\tan\tau\tanh\tau\right]^{-2}, \tag{22}$$

which is independent of energy.

It would also be interesting to see what happens when the rectangular forms of (12) and (15) are replaced by a smooth one. Instead of $\Delta'_3(x)$ let us consider

$$\Delta'_4(x) = \frac{1}{\lambda\epsilon}\left[G\left(x + \frac{\lambda\epsilon}{2}\right) - G\left(x - \frac{\lambda\epsilon}{2}\right)\right], \tag{23}$$

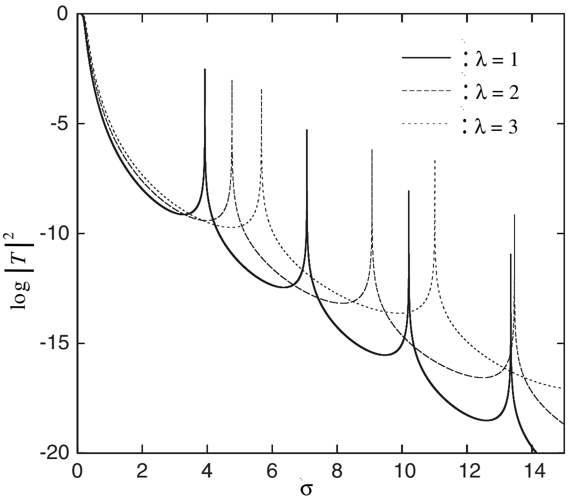

Figure 1. Transmission probability $|T|^2$ versus σ obtained with $V(x) = \sigma^2\Delta_3'(x)$ with $\lambda = 1, 2,$ and 3 and $\eta = 0.0005$.

where

$$G(x) = \frac{e^{-x^2/\epsilon^2}}{\sqrt{\pi}\epsilon}, \qquad \int_{-\infty}^{\infty} G(x)\,dx = 1. \tag{24}$$

The height and depth of the potential are again proportional to $1/\epsilon^2$. As a consequence, the $T(k)$ obtained by using $\Delta_4'(x)$ can be specified in terms of σ, λ and $\eta = k\epsilon$. In the limit of $\epsilon \to 0$ and $\eta \to 0$, $T(k)$ becomes independent of k. Christiansen *et al* argued that the partial transmission that they found is related to the fact that $\Delta_2'(x)$ that they used has a nonzero discontinuity at $x = 0$. Note however that $\Delta_4'(x)$ has no discontinuity at $x = 0$. We numerically solve the Schrödinger equation with potential $V(x) = \sigma^2\Delta_4'(x)$ and determine $T(k)$ and $R(k)$.

Figure 1 shows the logarithm of $|T|^2$ versus σ, obtained by using the rectangular $\Delta_3'(x)$ of (15). For λ and η we have chosen $\lambda = 1$, 2 and 3 and $\eta = 0.0005$. The results for $\lambda = 1$ are the same as their counterparts of [4]. In the limit of $\epsilon \to 0$, $|T|^2$ vanishes except when σ takes values σ_n $(n = 1, 2, \ldots)$ which correspond to the spikes of $|T|^2$ that are shown in figure 1. The σ_n increases as λ increases. Figure 2 shows the same quantities as those of figure 1 except that they have been obtained by using the smooth $\Delta_4'(x)$ of (23).

Figures 1 and 2 are very similar but there are differences. The λ dependence of σ_n is stronger in figure 1 than in figure 2. Even when $\lambda = 1$, figures 1 and 2 are slightly different. It is clear that the values of σ_n do depend on the shape of the assumed potential. In the case of $\lambda = 1$, the σ_n's for the rectangular potential and those for the smoothed version can be compared as follows:

$$\begin{aligned} &\sigma_1 = (3.927, 3.924, 4.044), &&\sigma_2 = (7.069, 7.065, 6.986), \\ &\sigma_3 = (10.210, 10.206, 9.895), &&\sigma_4 = (13.352, 13.346, 12.792). \end{aligned} \tag{25}$$

The three figures in the above parentheses mean the following. For example, σ_1 of the rectangular case is 3.927 in the limit of $\eta \to 0$ and it is 3.924 when $\eta = 0.0005$. For $\Delta_4'(x)$ we find $\sigma_1 = 4.044$ when $\eta = 0.0005$ and $\lambda = 1$. As the value of η becomes smaller, it becomes more difficult to solve the Schrödinger equation accurately. This is the reason why we do not have results for $\eta = 0$ of the smooth case.

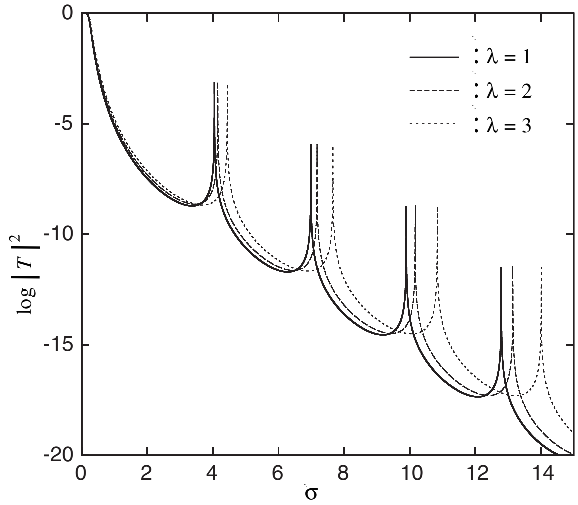

Figure 2. Transmission probability $|T|^2$ versus σ obtained with $V(x) = \sigma^2\Delta_4'(x)$ with $\lambda = 1, 2,$ and 3 and $\eta = 0.0005$.

Some of the features that we have examined are not restricted to the specific forms of $\Delta'(x)$ that we have assumed. This can be seen from the following general observations. In the transmission–reflection problem with an arbitrarily given potential (of any range), the transmission probability usually vanishes at threshold, i.e., $|T(k)|^2 \to 0$ as $k \to 0$. If there is a bound state or a resonance at threshold, however, there can be 'threshold anomaly' such that $|T(k)|^2$ remains finite as $k \to 0$ [5]. There are two types of the anomaly, I and II. We obtain $|T(0)|^2 = 1$ in type I, whereas $|T(0)|^2$ can take any value between 0 and 1 in type II. Type II can be found only if the potential is asymmetric as a function of x [5, 6]. When $f(\tau, \lambda) = 0$, there is a bound state or a resonance at threshold. Such situations that we have found for $\Delta_i'(x)$ $(i = 2, 3, 4)$ are illustrations of the threshold anomaly of type II.

Suppose we can control the range of the potential for which there is threshold anomaly. If the range is reduced to zero, the potential becomes one of the 'generalized point interactions' that can be characterized by the following boundary condition on the wavefunction at the origin:

$$\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}. \tag{26}$$

Here a, b, c and d, which are subject to constraint $ad - bc = 1$, and θ are all real parameters; see [3, 4] and references therein. Apart from θ that is unimportant [7], there are three independent parameters. We can put $\mathrm{e}^{\mathrm{i}\theta} = 1$ without losing generality. Then the transmission and reflection coefficients are given by

$$\begin{aligned} &T(k) = -2\mathrm{i}k/D(k), \qquad R(k) = -[c + \mathrm{i}k(a - d) - bk^2]/D(k), \\ &D(k) = c - \mathrm{i}k(a + d) + bk^2. \end{aligned} \tag{27}$$

This is for the case in which the incident wave is from the left; see (3). If the incident wave is from the right, $R(k)$ can be obtained by interchanging a and d in its expression given above [3]. The $T(k)$ does not depend on the direction of the incident wave.

There is a bound state of energy $E = -\kappa^2$ if $D(k) = 0$ for $k = \mathrm{i}\kappa$, where $\kappa \geqslant 0$ [3]. Assume that the zero energy bound state (and hence the threshold anomaly) is maintained in the process of reducing the range of the potential. The existence of the bound state with $\kappa = 0$ means that $c = 0$ and $ad = 1$. Furthermore the k-independence of the narrow width limit of $T(k)$ implies $b = 0$. If the potential is symmetric, we have $a = d = \pm 1$, $T(k) = 1$ and $R(k) = 0$. This is anomaly of type I. If the potential is asymmetric, we obtain $a \neq d$ and $T(k) < 1$. This is anomaly of type II. As can be seen from (27) with $b = c = 0$ this $T(k)$ is independent of k. The finite range potential with $\Delta_i'(x)$ $(i = 2, 3, 4)$ is antisymmetric. The anomaly of type II, however, can occur as long as the potential is asymmetric. The occurrence of partial transmission requires the potential be asymmetric but not necessarily antisymmetric.

Before ending we should mention two very recent papers [8] and [9] that are extensions of [4]. The $\Delta_2'(x)$ used in [4] is an odd function of x. This is a natural assumption in the sense that $\delta'(x)$ is usually considered as an odd function of x. In [8] a more general form that is not restricted to an odd function of x is considered. Interestingly enough, this generalization opens up the possibility of accommodating the entire family of point interactions that are represented by the boundary condition (26). In this communication, however, we have focused on $\Delta'(x)$ of the form of an odd function of x. The extension of [4] that we have presented is of a type that is different from but complementary to the one examined in [8]. In [9] the analyses of [4, 8] are extended to the second derivative of $\delta(x)$, which is beyond the scope of the present communication. Let us only mention that the threshold anomaly would also shed light on the analysis done in [9].

In summary, we examined the transmission–reflection problem with a potential of the form of $\Delta'(x)$ which, in the narrow width limit $\epsilon \to 0$, is reduced to $\delta'(x)$. With various forms of $\Delta'(x)$ we illustrated how the strength coefficient of the $\Delta'(x)$, which causes partial transmission, depends on the assumed form of $\Delta'(x)$. An important feature of these functions is that the heights and depths of the potential are proportional to $1/\epsilon^2$. It has the consequence that, in the narrow width limit of the potential, the transmission coefficient $T(k)$ becomes energy independent. We also discussed the problem in the light of threshold anomaly in which the partial transmission is related to the existence of a zero-energy bound state. Since $T(k)$ is energy independent as mentioned above, the threshold anomaly is extended to all energies.

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References

[1] Šeba P 1986 *Rep. Math. Phys.* **24** 111
[2] Patil SH 1994 *Phys. Scr.* **49** 645
[3] Coutinho F A B, Nogami Y and Perez J F 1997 *J. Phys. A: Math. Gen.* **30** 3937
[4] Christiansen P L, Arnbak H C, Zolotaryuk A V, Ermakov V N and Gaidei Y B 2003 *J. Phys. A: Math. Gen.* **36** 7589
[5] Senn P 1988 *Am. J. Phys.* **56** 916
[6] Nogami Y and Ross C K 1996 *Am. J. Phys.* **64** 923
[7] Coutinho F A B, Nogami Y and Perez J F 1999 *J. Phys. A: Math. Gen.* **32** L133
[8] Zolotaryuk A V, Christiansen P L and Iermakova S V 2006 *J. Phys. A: Math. Gen.* **39** 9329
[9] Zolotaryuk A V, Christiansen P L and Iermakova S V 2007 *J. Phys. A: Math. Gen.* **40** 5443